

Solving the Schwinger-Dyson equation for a scalar propagator in Minkowski space

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The Schwinger-Dyson equation for a scalar propagator is solved in Minkowski space with the help of an integral spectral representation, both for spacelike and timelike momenta. The equation is re-written into a form suitable for numerical solution by iterations. This procedure is described for a simple unphysical Lagrangian with a cubic interaction, with future extensions to more realistic theories in mind.

1. INTRODUCTION

The infinite tower of integral Schwinger-Dyson equations (SDEs) links n -point Green's functions ($n = 2, 3, \dots$) of a quantum-field theory. Their exact solution would provide complete information on the physics of the theory, including its non-perturbative regime. In practice, the system of SDEs has to be truncated and closed by making assumptions about the driving term, which contains more complicated Green's functions determined by equations that were thrown away. Most often only the simplest equation for the 2-point Green's function(s) (one particle propagator) is retained, its kernel contains n -point ($n = 3, \dots$) vertex the full form of which is unknown. Hence some physical ansatz has to be used and then the equations can be solved (usually numerically). One hopes that the solution provides some useful information on the behavior of the theory, in particular in the non-perturbative region. It is certainly interesting to compare the solution with results of alternative non-perturbative techniques [1].

In most papers on the solution of the SDE, the Wick rotation from Minkowski to Euclidean space is employed in order to escape singularities of the kernel inherent to physical Green functions. We instead attempt to find the solution directly in Minkowski space by making use of the spectral representation of the Green's functions based on their known (or assumed) analytical properties. In this contribution we will for simplicity illustrate this method of solution with a simple example of self-interacting scalar fields with super-renormalizable unphysical $g\Phi^3(x)$ coupling. Possibilities of extending the technique to more realistic theories are briefly discussed.

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2. FORMALISM

Our approach is a straightforward extension of the spectral decomposition method of Kusaka *et al.* [2], developed for the solution of the Bethe-Salpeter equation for scalar bound states. The Green's functions – one scalar particle propagator in our case – are written as spectral integrals over some weight functions. Then we put these parameterizations and an expression for the vertex function into the SDE, combine denominators with the help of the usual Feynman parameterization, integrate over the loop momentum/momenta and obtain the real integral equations for the weight functions depending on spectral variables. These equations are free of singularities and can be solved by iterations.

The generic spectral decomposition of the dressed renormalized scalar propagator reads:

$$G(p^2) = \int d\alpha \frac{\sigma(\alpha)}{p^2 - \alpha - i\epsilon}, \quad \sigma(\alpha) = \delta(\alpha - m^2) + \rho(\alpha), \quad (1)$$

where $\sigma(\alpha)$ is a positive spectral function. Here, we assume that σ has a singular contribution due to the propagation of physical particle with the mass m and a regular positive smooth part $\rho(\alpha)$ which starts at the two-particle threshold. This assumption means that the particle spectrum of the system is essentially perturbative, e.g., there is no confinement and also the contribution of possible bound states below $\alpha_{th} = 2m$ is neglected (the latter appears as corrections to the vertex from the poles in the 4-point Green's function).

The form of the SDE for an one particle propagator depends on the form of the interaction. For our toy model with $\mathcal{L}_{int} = -g\Phi^3(x)$ and introducing the renormalized (by on-mass shell subtraction) self-energy $\Pi_R(p^2)$ we get

$$G^{-1}(p^2) = Z(p^2 - m_0^2) - \Pi(p^2) = p^2 - m^2 - \Pi_R(p^2); \quad \Pi_R(m^2) = \frac{d}{dp^2}\Pi_R(m^2) = 0, \quad (2)$$

$$\Pi(p^2) = i \frac{Sg^2}{(2\pi)^4} \int d^4q \Gamma(p, q) G((p-q)^2) G(q^2), \quad (3)$$

where m_0 is a bare mass, the residue of $G(p^2)$ at the pole $p^2 = m^2$ equals 1; $S = 18$ is a combinatorial factor, Z is the renormalization function of the $\Phi(x)$ field, which is finite for the cubic interaction. The adopted renormalization procedure suggests the following spectral decomposition for $\Pi_R(p^2)$:

$$\Pi_R(p^2) = (p^2 - m^2)^2 \int d\alpha \frac{\rho_\pi(\alpha)}{p^2 - \alpha - i\epsilon}. \quad (4)$$

We will mostly work in the bare vertex approximation $\Gamma(p, q) = 1$. From the imaginary part of $G = G_0 + G_0\Pi_R G$ and from the dispersion relations of Eqs. (1) and (4) follows:

$$\rho(\omega) = \rho_\pi(\omega) + (m^2 - \omega) P \int_{4m^2}^{\infty} d\alpha \frac{\rho_\pi(\omega)\rho(\alpha) + \rho_\pi(\alpha)\rho(\omega)}{\alpha - \omega}. \quad (5)$$

From the SDE (3) we get the second relation between ρ_π and ρ :

$$\begin{aligned} \rho_\pi(\omega) = & \frac{Sg^2}{(2\pi)^4} \frac{1}{(\omega - m^2)^2} \left[\sqrt{1 - 4m^2/\omega} + \int d\alpha T(\alpha, m^2, \omega) \rho(\alpha) \right. \\ & \left. + \int \int d\alpha_1 d\alpha_2 T(\alpha_1, \alpha_2, \omega) \rho(\alpha_1) \rho(\alpha_2) \right], \end{aligned} \quad (6)$$

where $T(\alpha_1, \alpha_2, \omega)$ is a purely kinematical function which also determines the integration bounds. The set of equations (5,6) is solved by iterations.

The method works in essentially the same way also for more complicated theories. Of course, if there is more than one type of particles, one gets a set of coupled integral equations. In more complicated theories it might not be always possible to integrate out all Feynman parameters. Then kinematical functions analogous to $T(\alpha_1, \alpha_2, \omega)$ in Eq. (6) are expressed in terms of the remaining integrations. That might slow down and complicate the solution, in particular one has to determine numerically the bounds for integrals over the spectral parameters α .

3. RESULTS

The iterations work very well for small $\lambda = Sg^2/((2\pi)^4 m^2)$, but in that region the propagator is not far from the free one. With increasing λ the rate of convergence is slowing down rather fast, then for a certain value $\lambda_{crit} \sim 2.5$ it breaks down. We do not know at this stage the meaning of this critical value of the coupling constant, neither for our unphysical example of the scalar cubic interaction nor for other theories (e.g. Φ^4). We even do not know whether the SDE indeed does not have a (formal) solution for larger λ or whether it is just a failure of our numerical method. We would only observe that a very similar behavior was found in [3], where a similar coupling $\Phi^2\Psi$ was considered. In our case it is also possible (in a slightly modified formulation) to calculate the field renormalization function Z which is finite. It appears that very close to λ_{crit} (within 5 per cent) this function goes through zero (another sign of pathological behavior). We are aware of the fact that the Φ^3 model is in fact not defined at all [4], hence its study should be viewed as purely methodological and one has to be very cautious in making any generalization. It might be nevertheless interesting that when one includes one loop corrections to the vertex function, λ_{crit} drops by a factor of about 2.

Figure 1 shows the weight functions $\rho(\alpha)$ from below for $\lambda = 0.1, 0.5, 1.0, 2.2, 2.5$. The last line already shows wiggles, precursors of the numerical breakdown. Figure 2 displays the self-energies below threshold $s = 4m^2$ from above for $\lambda = 0.25, 1.0, 1.5, 2.2, 10.0$, compared with those obtained by Dyson (bubble) summations (for the last value $\lambda = 10.0$ we do not get a solution of the DSE). For small λ 's the Dyson summation approximates our full solutions rather well, for λ close to λ_{crit} they deviate by up to 25 per cent.

4. CONCLUSION

We have developed and tested in a cubic scalar toy model the method of solving DSE's in Minkowski space. Several applications of the method are in progress: studies of the scalar Φ^4 theory, of 3+1 QED in a quenched rainbow approximation, and of a theory with $\bar{\psi}\gamma_5\psi\Phi$ fermion-pseudoscalar coupling. The solution of SDEs in the described framework seems to be feasible for all of these models, even with more complicated structure of equations employed. When one tries to include more loops, much more painful algebraic manipulations has to be done to bring equations for weight functions into manageable form. We hope to prove by this further tests that this method is competitive, compared to solutions in Euclidean space.

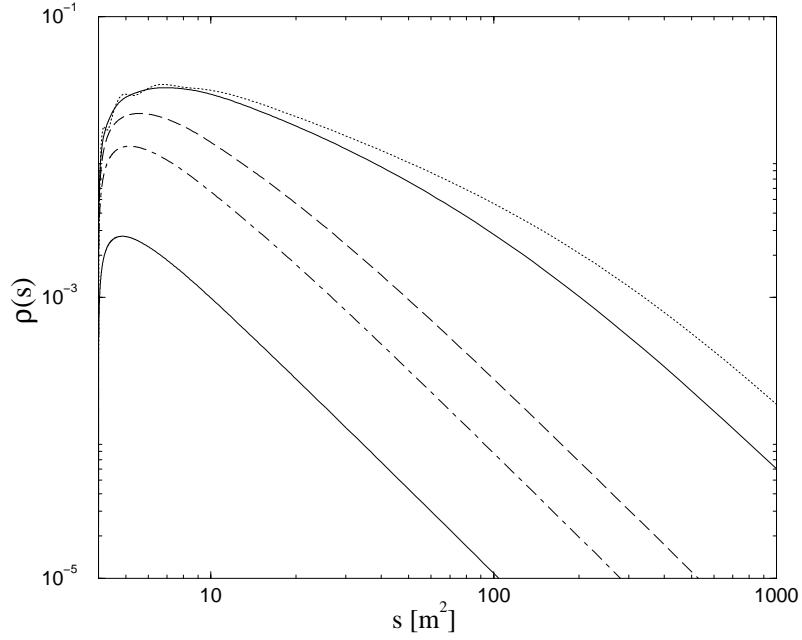


Fig.1

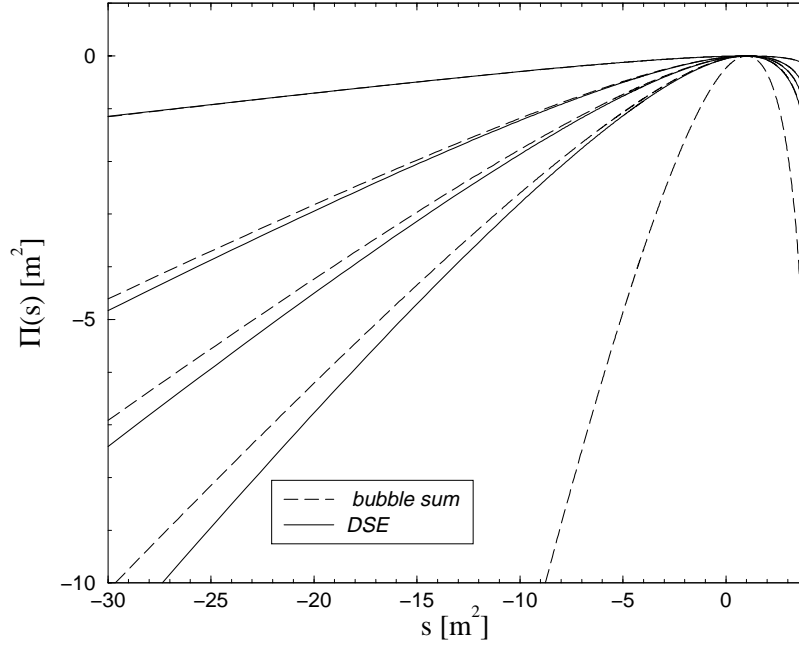


Fig. 2 Curves in figures are described in the text.

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